An Interior Penalty Nitsche method for diffusion problems

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Introduction

Toy model problem:

The Dirichlet boundary value problem :

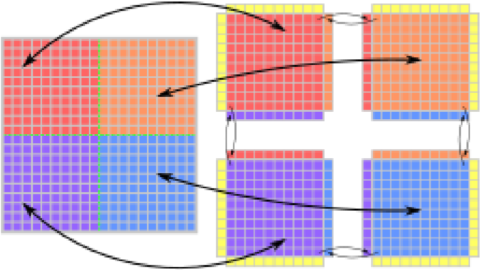
$$\nabla \cdot (-\boldsymbol{\kappa} \nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

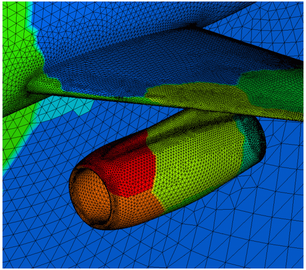
where $\boldsymbol{\kappa} : \Omega \to \mathbb{R}^{d \times d}$ (diffusivity) and $f : \Omega \to \mathbb{R}$ (source term).

Different numerical strategies for solving (1) : FE, FV, DG, MFE...

- Strong (cat. I) vs. Weak (cat. II) enforcement of Dirichlet BC,
- Practical interest of cat. II: Interface problems, Domain Decomposition methods,

Introduction





- Hot topic with vast literature : pioneering works can be traced back in 70's
  - [Nitsche, 1971] Weakly imposed via boundary penalty,
  - [Babuška, 1972] Weakly imposed via Lagrange Multipliers (BB condition),
  - [Barbosa & Hughes, 1991] Circumvent BB condition by adding stabilization terms,
  - [Stenberg, 1995] Closed connections between Barbosa *et al.* and Nitsche forms.

Motivation

- Re-interpretation of Babuška's strategy :

$$\underbrace{\lambda_h^{\text{old}} \rightarrow (\boldsymbol{\sigma} \cdot n) \quad \text{on} \quad \mathcal{F}_h^{\partial}}_{\text{Partial-hybridization}} \qquad \text{vs.} \qquad \underbrace{\lambda_h^{\text{new}} \rightarrow \text{Tr}(u) \quad \text{on} \quad \mathcal{F}_h}_{\text{Full-hybridization}}$$

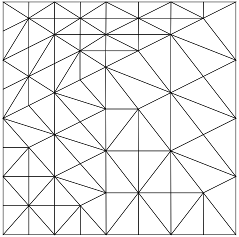
- Derivation of a stable Lagrange Multipliers formulation,
- Reduce computational costs by eliminating multipliers (explicitly):
  - (Strat. I) by static condensation or equiv.
  - (Strat. II) by algebraic transformation
- IPN method: a combination of interior/boundary penalty mechanisms

Outline

1. Fully-Hybridizable Nitsche Galerkin method
   - Intuitive derivation
   - Discrete formulation
   - Stability & convergence analysis
2. Free-multipliers version: IP Nitsche method
   - Static condensation vs Algebraic elimination
   - Equivalency
   - Stability & Convergence analysis
3. Numerical experiments (G. Etangsale)
4. Conclusion & Perspectives

I. Fully-Hybridizable Nitsche Galerkin method

- Mesh notation :
  - $\mathcal{T}_h$, the set of simplices covering Ω
  - $\mathcal{F}_h^i$, the set of interior interfaces
  - $\mathcal{F}_h^\partial$, the set of boundary interfaces
  - $\partial\mathcal{T}_h := \{\cup \partial E, \forall E \in \mathcal{T}_h\}$,



- The discrete approximation of u :
  - Interior variable : $u_h \in V_h \subset H^1(\Omega)$ s.t. $u_h|_E \in \mathbb{P}_k(E)$, for all $E \in \mathcal{T}_h$,
  - Skeleton variable : $\hat{u}_h \in \hat{V}_h^0 \subset L^2(\mathcal{F}_h)$ s.t. $\hat{u}_h|_F \in \mathbb{P}_k(F)$, for all $F \in \mathcal{F}_h$.
- Composite variable : $\boldsymbol{u}_h := (u_h, \hat{u}_h)$ and $\boldsymbol{V}_h := V_h \times \hat{V}_h^0$

I.a Intuitive derivation

Let $\boldsymbol{v}_h := (v_h, \hat{v}_h) \in \boldsymbol{V}_h$, u, the exact solution and $\boldsymbol{\sigma} := -\boldsymbol{\kappa}\boldsymbol{\nabla} u$, its corresponding flux.

- Integrating by part and summing over all mesh element:

$$(\nabla \cdot \boldsymbol{\sigma}, v_h)_{\mathcal{T}_h} = -(\boldsymbol{\sigma}, \boldsymbol{\nabla} v_h)_{\mathcal{T}_h} + \langle \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, v_h \rangle_{\partial\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}$$

- Using the continuity requirement :

$$\langle \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \hat{v}_h \rangle_{\partial\mathcal{T}_h} = 0$$

- Combining these equations :

$$-(\boldsymbol{\sigma}, \boldsymbol{\nabla} v_h)_{\mathcal{T}_h} + \langle \hat{\boldsymbol{\sigma}}, [\![\boldsymbol{v}_h]\!] \rangle_{\partial\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}, \tag{2}$$

where $[\![\boldsymbol{v}_h]\!] := (v_h - \hat{v}_h)\boldsymbol{n}$.

I.a How to build a discrete version

A discrete analog of (2):

$$-\left(\boldsymbol{\sigma}_h(\boldsymbol{u}_h), \nabla v_h\right)_{\mathcal{T}_h} + \langle \hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h), [\![\boldsymbol{v}_h]\!] \rangle_{\partial\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}\,, \tag{3}$$

Question: How to choose $\boldsymbol{\sigma}_h(\boldsymbol{u}_h)$ and $\hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h)$?

- Via a consistency argument, i.e., $\boldsymbol{\sigma}_h(\boldsymbol{u}) = \hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}) = \boldsymbol{\sigma}$, then

$$\boldsymbol{\sigma}_h(\boldsymbol{u}_h) := -\boldsymbol{\kappa}\nabla u_h \quad \text{in } \mathcal{T}_h \quad \text{and} \quad \hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h) := \boldsymbol{\sigma}_h(u_h) + \tau([\![\boldsymbol{u}_h]\!]) \quad \text{on } \partial\mathcal{T}_h$$

where $\tau : L^2(\partial\mathcal{T}_h) \to \mathbb{R}^+$ is a penalty function. For all $E \in \mathcal{T}_h$ and $F \in \partial E$,

$$\tau|_{E,F} := \varpi_0 t_{E,F}, \tag{4}$$

where $\varpi_0 > 0$, a given parameter and $t_{E,F} := \kappa_{E,F}/h_E$, transmissibility.

I.b Discrete weak problem

Flux formulation:

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$a_h^{(\epsilon)}(\boldsymbol{u}_h, \boldsymbol{v}_h) = (f, v_h)_{\mathcal{T}_h} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \tag{5}$$

where $\epsilon \in \{0, \pm 1\}$ and the bilinear form $a_h^{(\epsilon)}$ is given by

$$a_h^{(\epsilon)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := -(\boldsymbol{\sigma}_h(u_h), \boldsymbol{\nabla} v_h)_{\mathcal{T}_h} + \underbrace{\langle \hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h), [\![\boldsymbol{v}_h]\!]\rangle_{\partial\mathcal{T}_h}}_{\text{Consistency+Penalty}} + \epsilon \underbrace{\langle \boldsymbol{\sigma}_h(v_h), [\![\boldsymbol{u}_h]\!]\rangle_{\partial\mathcal{T}_h}}_{\text{Symmetry}}$$

Variants of FHN methods

Incomplete ($\epsilon = 0$), Symmetric ($\epsilon = +1$), and Nonsymmetric ($\epsilon = -1$).

I.c Stability and convergence analysis

Let us introduce mesh-dependent norm (Stability norm):

$$|\!|\!| \boldsymbol{u}_h |\!|\!|_h^2 := \| \boldsymbol{\kappa}^{\frac{1}{2}} \nabla u_h \|_{\mathcal{T}_h}^2 + |\boldsymbol{u}_h|_{\frac{1}{2},h}^2 \quad \text{where} \quad |\boldsymbol{u}_h|_{\frac{1}{2},h}^2 := \| \mathfrak{t}^{\frac{1}{2}} [\![\boldsymbol{u}_h]\!] \|_{\partial \mathcal{T}_h}^2 \tag{6}$$

- Coercivity: For all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ then $a_h^{(\epsilon)}(\boldsymbol{v}_h, \boldsymbol{v}_h) \gtrsim |\!|\!| \boldsymbol{v}_h |\!|\!|_h^2$ assuming $\varpi_0 \geq \varpi_\sharp$,
- Boundedness: For all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\boldsymbol{w} \in \boldsymbol{V} \oplus \boldsymbol{V}_h$ then

$$a_h^{(\epsilon)}(\boldsymbol{w}, \boldsymbol{v}_h) \lesssim |\!|\!| \boldsymbol{w} |\!|\!|_{h,*} |\!|\!| \boldsymbol{v}_h |\!|\!|_h$$

where $|\!|\!| \boldsymbol{w} |\!|\!|_{h,*}^2 := |\!|\!| \boldsymbol{w} |\!|\!|_h^2 + \sum_{E \in \mathcal{T}_h} h_E \| \boldsymbol{\kappa}^{\frac{1}{2}} \nabla w \cdot n \|_{\partial E}^2$ (Continuity norm)

Optimal error estimate in stability-norm

$$|\!|\!| \boldsymbol{u} - \boldsymbol{u}_h |\!|\!|_h^2 \leq O(h^k) \tag{7}$$

Some issues of FHN methods:

Let U_h and $\hat{\mathrm{U}}_h$ be the set of dofs of u_h and $\hat{u}_h$. The algebraic linear system :

$$\begin{array}{lcl} a_h^{(\epsilon)}(\boldsymbol{u}_h,(v_h,0)) & : & \mathbb{A}_{uu}\mathrm{U}_h + \mathbb{A}_{u\hat{u}}\hat{\mathrm{U}}_h = \mathrm{F}, \\ a_h^{(\epsilon)}(\boldsymbol{u}_h,(0,\hat{v}_h)) & : & \mathbb{A}_{\hat{u}u}\mathrm{U}_h + \mathbb{A}_{\hat{u}\hat{u}}\hat{\mathrm{U}}_h = 0. \end{array} \tag{8}$$

The global size of (8) : dim $\boldsymbol{V}_h = \dim V_h + \dim \hat{V}_h^0$ observing that

$$\dim V_h << \dim \hat{V}_h^0 \quad \text{since} \quad V_h \subset H^1(\mathcal{T}_h) \cap C^0(\bar{\Omega}) \quad \text{and} \quad \hat{V}_h^0 \subset L^2(\mathcal{F}_h)$$

A cost-effective strategy : an explicit elimination of $\hat{\mathrm{U}}_h$ (Strat. I) or $\hat{u}_h$ (Strat. II)

II.a Explicit elimination of $\hat{\mathrm{U}}_h$

Strategy 1: Shur complement strat.

- The key observation : $\mathrm{A}_{\hat{u}\hat{u}}$ inherits a block-diagonal structure $(\hat{V}_h^0 \subset L^2(\mathcal{F}_h))$,
- All computations, such as matrix inversion, can be performed edgewise,

Thus, we infer the reduced linear system:

$$[\mathbb{A}_{uu} - \mathbb{A}_{u\hat{u}}\mathbb{A}_{\hat{u}\hat{u}}^{-1}\mathbb{A}_{\hat{u}u}]\mathbb{U}_h = \mathrm{F}. \quad (9)$$

The matrix on the left-hand side of (9) is called the Schur complement of $\mathbb{A}_{\hat{u}\hat{u}}$.

- Schur complement $\Rightarrow$ Additional computational complexity,
- Construction of interaction matrices and its inverse : $\mathbb{A}_{u\hat{u}}$, $\mathbb{A}_{\hat{u}\hat{u}}^{-1}$ and $\mathbb{A}_{\hat{u}u}$

II.a Explicit elimination of $\hat{u}_h$

$$a_h^{(\epsilon)}(\boldsymbol{u}_h, \boldsymbol{v}_h) := -(\boldsymbol{\sigma}_h(u_h), \nabla v_h)_{\mathcal{T}_h} + \underbrace{\langle \hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h), [\![\boldsymbol{v}_h]\!]\rangle_{\partial\mathcal{T}_h}}_{\text{Consistency+Penalty}} + \epsilon \underbrace{\langle \boldsymbol{\sigma}_h(v_h), [\![\boldsymbol{u}_h]\!]\rangle_{\partial\mathcal{T}_h}}_{\text{Symmetry}}$$

Strategy 2: Ghost-hybridization

- Eliminate $\hat{u}_h$ from skeletal contributions :

$$\langle \hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h), [\![\boldsymbol{v}_h]\!]\rangle_{\partial\mathcal{T}_h} \quad \text{and} \quad \langle \boldsymbol{\sigma}_h(v_h), [\![\boldsymbol{u}_h]\!]\rangle_{\partial\mathcal{T}_h}$$

- To do that, we use the **conservativity** requirement i.e., $\langle \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \hat{v}_h\rangle_{\partial\mathcal{T}_h} = 0$:

$$\hat{u}_h = \begin{cases} u_h + \varrho[\![\boldsymbol{\sigma}_h(u_h)]\!] \\ 0 \end{cases} \quad \text{and} \quad \hat{\boldsymbol{\sigma}}_h = \begin{cases} \{\boldsymbol{\sigma}_h(u_h)\}^{\boldsymbol{\omega}} & \text{on} \quad \mathcal{F}_h^i, \\ \boldsymbol{\sigma}_h(u_h) + \tau u_h \boldsymbol{n} & \text{on} \quad \mathcal{F}_h^\partial, \end{cases} \tag{10a}$$

where $\varrho|_F := 1/(\tau_{E_1,F} + \tau_{E_2,F})$ and $\boldsymbol{\omega}|_F := (\varrho|_F \tau_{E_1,F}, \varrho|_F \tau_{E_2,F})$.

II.b Variational form of IPN method

Theorem "Equivalency of FHN and IPN"

Let $\boldsymbol{u}_h := (u_h, \hat{u}_h) \in \boldsymbol{V}_h$ be the solution of (5). Then, $u_h \in V_h$ also satisfies

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that,} \\ r_h^{(\epsilon)}(u_h, v_h) = (f, v_h)_{\mathcal{T}_h}, \quad \forall v_h \in V_h, \end{cases} \tag{11}$$

where $a_h : V_h \times V_h \to \mathbb{R}$ is given as follows,

$$r_h^{(\epsilon)}(u_h, v_h) := -(\boldsymbol{\sigma}_h(u_h), \boldsymbol{\nabla} v_h)_{\mathcal{T}_h} + \epsilon s_h(u_h, v_h) + \eta_h(u_h, v_h). \tag{12}$$

Here, s_h and $\eta_h^{(\epsilon)}$ are the interior penalty and Nitsche's boundary penalty mechanisms,

$$s_h(u_h, v_h) := -\langle \varrho [\![\boldsymbol{\sigma}_h(u_h)]\!], [\![\boldsymbol{\sigma}_h(v_h)]\!] \rangle_{\mathcal{F}_h^i},$$

$$\eta_h^{(\epsilon)}(u_h, v_h) := \langle \boldsymbol{\sigma}_h(u_h) \cdot \boldsymbol{n}, v_h \rangle_{\mathcal{F}_h^\partial} + \epsilon \langle \boldsymbol{\sigma}_h(v_h) \cdot \boldsymbol{n}, u_h \rangle_{\mathcal{F}_h^\partial} + \langle \tau u_h, v_h \rangle_{\mathcal{F}_h^\partial}$$

Proof of Theorem (straightforward)

- Considering that $[\![\hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h)]\!]|_F = 0$ on $F \in \mathcal{F}_h^i$ and $\hat{v}_h|_F = 0$ on $F \in \mathcal{F}_h^\partial$,

$$\langle \hat{\boldsymbol{\sigma}}_h(\boldsymbol{u}_h), [\![\boldsymbol{v}_h]\!] \rangle_{\partial \mathcal{T}_h} = \langle \hat{\boldsymbol{\sigma}}_h(u_h) \cdot \boldsymbol{n}, v_h \rangle_{\mathcal{F}_h^\partial}$$

- Using the definition of $\hat{u}_h$, the symmetry term can be rewritten as follows;

$$\langle \boldsymbol{\sigma}_h(v_h), [\![\boldsymbol{u}_h]\!] \rangle_{\partial \mathcal{T}_h} = \langle \boldsymbol{\sigma}_h(v_h) \cdot \boldsymbol{n}, u_h \rangle_{\mathcal{F}_h^\partial} - \langle \varrho [\![\boldsymbol{\sigma}_h(u_h)]\!], [\![\boldsymbol{\sigma}_h(v_h)]\!] \rangle_{\mathcal{F}_h^i}.$$

II.c Stability and convergence analysis of IPN

Let us introduce mesh-dependent norm (Stability norm):

$$\|u_h\|_h^2 := \|\boldsymbol{\kappa}^{\frac{1}{2}} \nabla u_h\|_{\mathcal{T}_h}^2 + |u_h|_\partial^2 \quad \text{where} \quad |u_h|_\partial^2 := \|t^{\frac{1}{2}} u_h\|_{\mathcal{F}_h^\partial}^2 \tag{14}$$

- Coercivity: For all $v_h \in V_h$ then $r_h^{(\epsilon)}(v_h, v_h) \gtrsim \|v_h\|_h^2$ assuming $\varpi_0 \geq \varpi_\sharp$,
- Boundedness: For all $v_h \in V_h$ and $w \in V \oplus V_h$ then

$$r_h^{(\epsilon)}(w, v_h) \lesssim \|w\|_{h,*} \|v_h\|_h$$

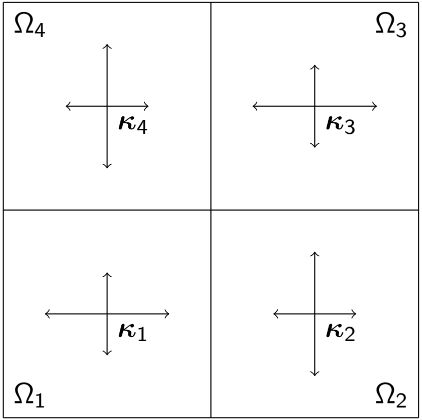
where $\|w\|_{h,*}^2 := \|w\|_h^2 + \sum_{E \in \mathcal{T}_h} h_E \|\boldsymbol{\kappa}^{\frac{1}{2}} \nabla w \cdot n\|_{\partial E}^2$ (Continuity norm)

Error estimate in L^2-norm (via Aubin-Nitsche lift)

$$\|u - u_h\|_{\mathcal{T}_h} \leq O(h^{k+1} + |\epsilon - 1|\, h^{k+\frac{1}{2}}). \tag{15}$$

Convergence is optimal iff $\epsilon = 1$ and sub-optimal (half-order only) else

III. Numerical experiments



- $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 = [0,1]^2$,
- Heterogeneous/Anisotropic diffusion tensor $\boldsymbol{\kappa}$:

$$\boldsymbol{\kappa}_{1,3}(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \qquad \text{for } (x,y) \in \Omega_{1,3},$$

$$\boldsymbol{\kappa}_{2,4}(x,y) = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{for } (x,y) \in \Omega_{2,4}.$$

  with $\varepsilon \in \{1 \text{ , } 10^2 \text{ and } 10^6\}$.

- Exact solution in Ω : $u(x,y) = \sin(\pi x)\sin(\pi y)$

Convergence analysis

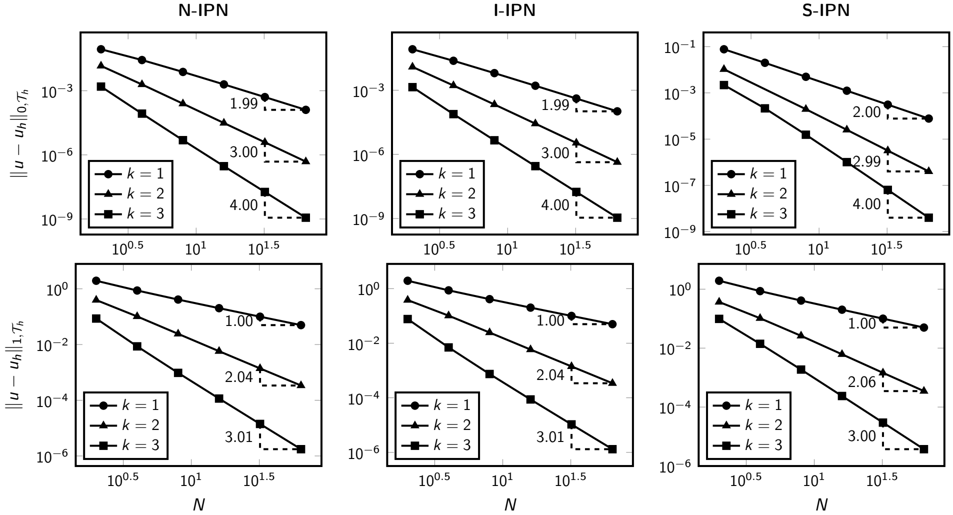
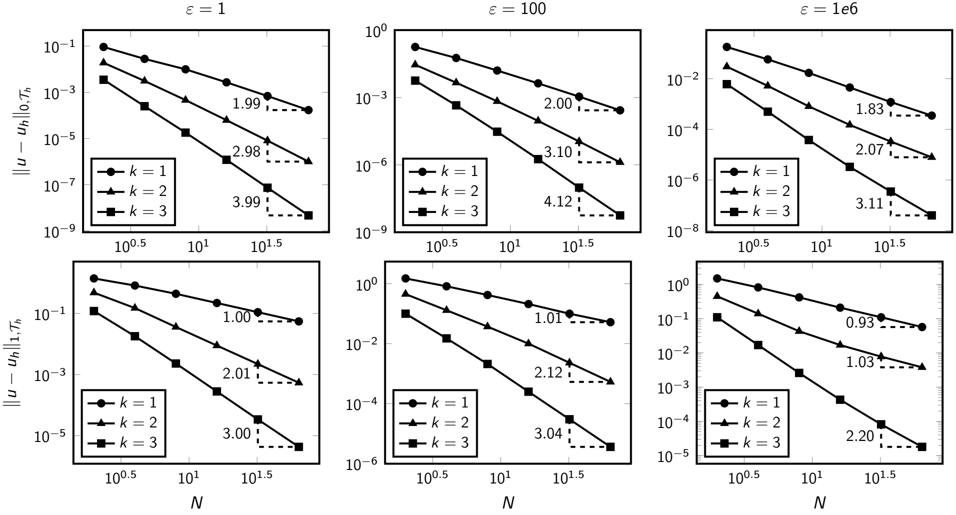


Figure: History of convergence in the L^2- and H^1-norm estimate versus the mesh-size $h = 1/N$.

Test 1: Influence of the interior penalty



Estimated convergence rates of the Non-symetric IP Nitsche variant ($\epsilon = 1, 10^2, 10^6$)

Conclusion

Novel class of IP Nitsche method for elliptic problems :

- Well-posedness
- Numerical traces on skeleton
- Optimal (estimated) convergence rates
- Extension to linear elasticity and stokes problem.

To explore :

- $u_h^* =$ Post-processing of $(u_h, \hat{u}_h)$:

$$\| u - u_h^* \|_{\mathrm{L}_2} \leq O(h^{k+2}) \qquad \text{(superconvergence)}$$

- $\boldsymbol{\sigma}_h^* =$ H(div)-conform reconstruction of $(\boldsymbol{\sigma}_h, \hat{\boldsymbol{\sigma}}_h)$:

$$\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^* \|_{\mathrm{div}} \leq O(h^{k+1})$$