Conjecture of Lehmer and Rényi numeration system in variable basis

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The search for very large prime numbers has a long history. The method of linear recurrence sequences of integers (Δ_m), typically satisfying

$$\Delta_{m+n+1} = A_1 \Delta_{m+1} + A_2 \Delta_{m+2} + \ldots + A_n \Delta_{m+n},$$

in which prime numbers can be found, has been investigated from several viewpoints : Bilu, Hanrot, Voutier '01 ; Everest, Stevens, Tamsett, Ward '07 ; Everest, van der Poorten, Shparlinski, Ward '03.

For some m, Δ_m is $=$ a prime p.

In 1933, given a monic integer polynomial $P(x) = \prod_{i=1}^{d}(x - \alpha_i)$, Lehmer developped an exhaustive approach from the Pierce numbers of P :

$$\Delta_n = \Delta_n(P) = \prod_{i=1}^{d}(\alpha_i^n - 1).$$

The sequence (A_i) is then the coefficient vector of the integer monic polynomial which is the least common multiple of the $d+1$ polynomials :

$$P_{(0)}(x) = x - 1,$$

$$P_{(1)}(x) = \prod_{i=1}^{d}(x - \alpha_i), P_{(2)}(x) = \prod_{i>j=1}^{d-1}(x - \alpha_i\alpha_j), \ldots, P_{(d)}(x) = x - \alpha_1\alpha_2 \ldots \alpha_d.$$

Large prime numbers, possibly at a certain power, can be found in the factorizations of $|\Delta_n|$ that have large absolute values (Dubickas 2011, Ji and Qin 2015, in connection with Iwasawa theory). This can be done fairly quickly if the absolute values $|\Delta_n|$ do not increase too rapidly (slow growth rate).

If P has no root on the unit circle, Lehmer proves

$$\lim_{n\to\infty} \frac{\Delta_{n+1}}{\Delta_n} = \mathrm{M}(P).$$

Einsiedler, Everest and Ward '00 revisited and extended the results of Lehmer in terms of the dynamics of toral automorphisms :

They considered expansive (no root on $|z| = 1$), ergodic (no α_i is a root of unity) and quasihyperbolic (if P is ergodic but not expansive) polynomials P and number theoretic heuristic arguments for estimating

the densities of primes in (Δ_n).

In the quasihyperbolic case (for instance for irreducible Salem polynomials P), the previous convergence does not extend but the following more robust convergence law holds :

$$\lim_{n\to\infty} \Delta_n^{1/n} = \mathrm{M}(P)$$

and

if P has a small Mahler measure, $< \Theta = 1.32\ldots$, it is reciprocal, the quotients Δ_n/Δ_1 are perfect squares for all $n \geq 1$ odd. Then, with $\Gamma_n(P) := \sqrt{\Delta_n/\Delta_1}$ in such cases, they obtain the existence of the limit

$$\lim_{j\to\infty} \frac{j}{\mathrm{Log}\,\mathrm{Log}\,\Gamma_{n_j}}, \tag{1}$$

(n_j) being a sequence of integers for which Γ_{n_j} is prime, as a consequence of Merten's Theorem. This limit, say E_P, is likely to satisfy the inequality : $E_P \geq 2e^{\gamma}/\mathrm{Log}\,\mathrm{M}(P)$, where $\gamma = 0.577\ldots$ is the Euler constant. Moreover, by number-fields analogues of the heuristics for Mersenne numbers (Wagstaff, Caldwell), they suggest that the number of prime values of $\Gamma_{n_j}(P)$ with $n_j \leq x$ is approximately

$$\frac{2e^{\gamma}}{\mathrm{Log}\,\mathrm{M}(P)} \mathrm{Log}\, x. \tag{2}$$

This result shows the interest of having a polynomial P of small Mahler measure to obtain a sequence (Δ_n) associated with P very rich in primes.

- still obscure, no theory, and reflects the deep arithmetics of the factorization of the integers $|\Delta_n|$ and of the quantities Γ_n.

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Definition : Weil height : let $\alpha \in \overline{\mathbb{Q}}^*$, $P_\alpha(X) = a_0(X-\alpha_1)(X-\alpha_2)\ldots(X-\alpha_n)$ $= a_0X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n \in \mathbb{Z}[X],\ a_0a_n \neq 0$, its minimal polynomial.
The (abs. log.) Weil height of α is

$$h(\alpha) = \frac{1}{n}\mathrm{Log}\left(|a_0|\prod_{i=1}^{n}\max\{1,|\alpha_i|\}\right)$$

Prop : $h(p/q) = \mathrm{Log}\max(|p|,|q|)$, $(p,q)=1$, $\quad h(1)=0$,
$h(\alpha) \geq 0$ for all $\alpha \in \overline{\mathbb{Q}}^*$,
$h(\alpha^r) = |r|h(\alpha)$, for $r \in \mathbb{Z}$, $\alpha \in \overline{\mathbb{Q}}^*$, $\quad h(1/\alpha) = h(\alpha)$,
$h(\sigma(\alpha)) = h(\alpha)$, for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Definition : Mahler measure : for

$$P(X) = a_0(X-\alpha_1)(X-\alpha_2)\ldots(X-\alpha_n) =$$
$$a_0X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n \in \mathbb{Z}[X], \qquad a_0 a_n \neq 0$$

then

$$\mathrm{M}(P) := |a_0| \prod_{i, |\alpha_i| \geq 1} |\alpha_i| \qquad \geq 1.$$

multiplicativity : $P = P_1 \times P_2 \times \ldots \times P_m, \Rightarrow \mathrm{M}(P) = \mathrm{M}(P_1)\ldots\mathrm{M}(P_m)$.

ex. : $P = \Phi_1 \times \ldots \times \Phi_r \times R$ with R irr. pol., Φ_j cyclot. $\Longrightarrow \mathrm{M}(P) = \mathrm{M}(R)$.

α alg. number, deg $\alpha = n$, P_α his minimal polynomial, $\mathrm{M}(\alpha) := \mathrm{M}(P_\alpha)$.
Absolute logarithmic, Weil height of α :

$$h(\alpha) := \frac{\mathrm{Log}\,\mathrm{M}(\alpha)}{d}$$

facts : $\mathrm{M}(\alpha) = \mathrm{M}(\alpha^{-1})$,

$\mathrm{M}(\alpha) = \alpha$ if $\alpha \in \mathrm{S}$ (= set of Pisot numbers ; $|\alpha_i| < 1$),

$\mathrm{M}(\alpha) = \alpha$ if $\alpha \in \mathrm{T}$ (= set of Salem numbers ; $|\alpha_i| < 1$ with at least one $|\alpha_j| = 1$),

$\mathrm{M}(\alpha) = 1$ if α is a root of unity.

Kronecker's Theorem(1857) : Let α be a nonzero algebraic integer. Then $\mathrm{M}(\alpha) = 1$ iff α is a root of unity.

$$\mathrm{M}(\alpha) = 1 \qquad : \qquad \alpha \text{ root of unity}$$

$$\mathrm{M}(\tau) = \tau = 1.176280\ldots \qquad : \qquad \tau \text{ Lehmer's number}$$

$$\tau = \text{dominant root of } X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

In between, in $(1, 1.176280\ldots)$: no example of $\mathrm{M}(\alpha)$ known.

Adler Marcus (1979) (topological entropy and equivalence of dynamical systems), Perron-Frobenius theory) :

$$\{\mathrm{M}(\alpha) \mid \alpha \text{ alg. number}\} \subset \mathbb{P}_{Perron},$$

$$\{\mathrm{M}(P) \mid P \in \mathbb{Z}[X]\} \subset \mathbb{P}_{Perron}.$$

Two strict inclusions (Dubickas 2004, Boyd 1981).

Definition : $\mathbb{P}_{Perron} := \{1\} \cup$
$\{\alpha >$ 1 is a real algebraic integer,for which the conjugates $\alpha^{(i)}$ satisfy $|\alpha^{(i)}| < \alpha\}$

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Conjecture of Lehmer : **there exists** $c > 0$ **such that**

$$\mathrm{M}(\alpha) \geq 1 + c$$

for any algebraic number $\alpha \neq 0$ **which is not a root of unity**,

i.e. the interval $(1, 1 + c) \cap \mathbb{P}_{Perron}$ is deprived of any value of Mahler measure of any algebraic number.

$->$ values : discontinuity at 1 (meaning, sense, of c ?).

Lehmer's problem (1933)

in the exhaustive search for large prime numbers : *if* ε *is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \ldots + a_r$$

where the a_i*s are integers, such that the absolute value of the product of those roots of* f *which lie outside the unit circle, lies between* 1 *and* $1 + \varepsilon$*... Whether or not the problem has a solution for* $\varepsilon < 0.176$ *we do not know*.

Lehmer's number $= 1.176280$.

Northcott's Theorem : for all $B \geq 0$, $d \geq 1$,

$$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < +\infty.$$

At bounded degree : finite number of algebraic integers.

For proving the Conjecture : tests on an infinite number of algebraic integers β in a neighbourhood of 1.

The usual degree of β is tending to infinity chaotically if $|\beta| > 1$ tends to 1. It is not a good variable, will be changed to another degree, called *dynamical degree of β*.

Lehmer's Problem is a limit problem + restrictions :

$$\mathrm{M}(P) := |a_0| \prod_{i,|\alpha_i|\geq 1} |\alpha_i| \quad \Longrightarrow \quad \mathrm{M}(P) := |a_0| \geq |a_0|.$$

Let $\alpha \in \overline{\mathbb{Q}},\ P = P_\alpha$:

- $*$ if $\alpha \in \overline{\mathbb{Q}} \setminus \mathscr{O}_{\overline{\mathbb{Q}}}$, then $|a_0| \geq 2 \quad \Longrightarrow \quad \mathrm{M}(P) \geq 2$,
- $*$ if α is an algebraic integer which is not reciprocal ($P_\alpha \neq P^*_\alpha$ with

$$P^*_\alpha(X) = X^{\deg P_\alpha} P_\alpha(1/X) \quad),$$

  Smyth's Theorem '71 $\Longrightarrow$ $\mathrm{M}(P_\alpha) \geq \Theta = 1.32\ldots$ (= smallest Pisot number, $X^3 - X - 1$ mini. pol.).

- $*$ restriction to real reciprocal algebraic integers is sufficient : if $\alpha \in \mathscr{O}_{\overline{\mathbb{Q}}}$, which is reciprocal ($P_\alpha = P^*_\alpha$), consider its house

$$\max\{|\alpha_i|\} =: \overline{|\alpha|} \in \mathscr{O}_{\overline{\mathbb{Q}}}$$

  which is real, ≥ 1.

Attack (context) : $\beta > 1$ **real** reciprocal algebraic integer,

$$1 < \beta \leq \boxed{\beta} \leq \mathrm{M}(\beta).$$

β tends to 1^+,
$\boxed{\beta}$ tends to 1^+.
? minimum of $\beta \to \mathrm{M}(\beta)$, of $\boxed{\beta} \to \mathrm{M}(\boxed{\beta})$.

assumption : existence of a real reciprocal algebraic integer $\beta > 1$ having : $\mathrm{M}(\beta) < 1.176280\ldots$ Lehmer's number.

define $\mathrm{dyg}(\beta)$ new integer.

asymptotic expansions of the poles of the dynamical zeta function $\zeta_\beta(z)$ as a function of $\mathrm{dyg}(\beta)$.

Previously : Dobrowolski's inequality ('79) : for any reciprocal algebraic integer α of degree d,

$$\mathrm{M}(\alpha) > 1 + (1-\varepsilon)\left(\frac{LogLog\ d}{Log\ d}\right)^3, \qquad d > d_1(\varepsilon).$$

(Dobrowolski, $1/1200$, Schinzel, $1-\varepsilon$ for $d > d_1$)
here,
the lower bound in the rhs tends to 1 when d tends to infinity. Useless.

Remarkable inequality. Not satisfying. Need to improve Dobrowolski's inequality by another method.

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Why use dynamical zeta functions $\zeta_\beta(z)$ of the β-shift ? (flash in 2009)

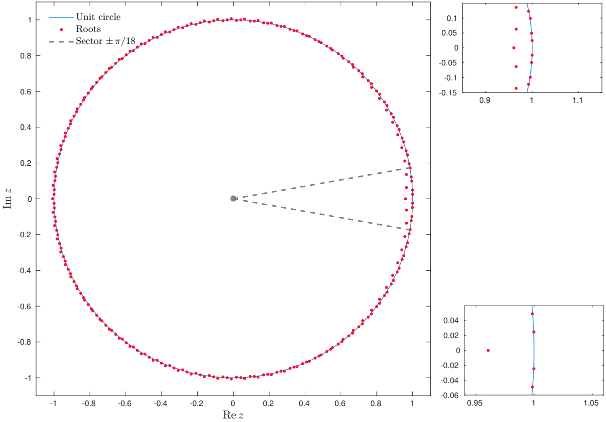
Because the Conjecture of Lehmer can be proved (VG, 2016, UDT) by this means for β running over

$\{\theta_n^{-1}; n \geq 3\}$, (seq. of Perron numbers > 1 tending to 1)

where
θ_n is the unique root of $-1+x+x^n$ in $(0,1)$.

- $*$ the information lies in the poles inside $D(0,1)$ the open unit disk,
- $*$ the poles of modulus < 1 form a **lenticulus.**

goal : extend the method to all βs > 1 tending to 1.



Unit circle
Roots
Sector $\pm\pi/18$
Im z
Re z
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0.1
0.05
-0.05
-0.1
-0.15
0.9
1.1
0.04
0.02
-0.02
-0.04
-0.06
0.95
1.05

Let $1 < \beta < (1+\sqrt{5})/2$ be **a real number.**

• Consider the dynamical system :

$$(X = [0,1], B, T_\beta, \mu)$$

where

$$T_\beta : \ x \mapsto \beta x \quad (\bmod 1) = \{\beta x\}, \qquad [0,1] \to [0,1].$$

is the β-transformation.
Invariant measure (Parry, '60) abs cont Lebesgue, pure jump, density :

$$h_\beta(x) = C \sum_{n, x < T_\beta^n(1)} \frac{1}{\beta^{n+1}}$$

unique (Rényi, '57), ergodic (Parry, '60), maximal (Hofbauer, '78)

• Later on, we will specialize $\beta > 1$ to run over reciprocal algebraic integers.

The "second" analytic function uniquely associated with β :

$$(X, B, T, \mu) = ([0,1],\ T_\beta,\ h_\beta(x)dx)$$

Theorem

Let $\beta \in (1, \theta_2^{-1})$. Then, the Artin-Mazur dynamical zeta function

$$\zeta_\beta(z) := \exp\Big(\sum_{n=1}^{\infty} \frac{\#\{x \in [0,1] \mid T_\beta^n(x) = x\}}{n} z^n\Big), \qquad (3)$$

counting the number of periodic points of period dividing n, is nonzero and meromorphic in $\{z \in \mathbb{C} : |z| < 1\}$, and such that $1/\zeta_\beta(z)$ is holomorphic in $\{z \in \mathbb{C} : |z| < 1\}$,

Theorem (Takahashi, Ito-Takahashi)

Let $\beta \in (1, \theta_2^{-1})$ be a real number. Then

$$\zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z)\left(\sum_{n=0}^{\infty} T_\beta^n(1)\, z^n\right)} \tag{4}$$

where N is the minimal positive integer such that $T_\beta^N(1) = 0$; in the case where $T_\beta^j(1) \neq 0$ for all $j \geq 1$, "z^N" has to be replaced by "0".

$$-1 + t_1 z + t_2 z^2 + t_3 z^3 + \ldots = f_\beta(z) = -(1 - \beta z)\left(\sum_{n=0}^{\infty} T_\beta^n(1)\, z^n\right),$$

Coefficients in $\{-1, 0, +1\}$, finite alphabet : Carlson-Polya dichotomy.

Up to the sign, the denominator is the Parry Upper function $f_\beta(z)$ at β (Generalized Fredholm Determinant in Theory of Operators). It satisfies

$$(i) \quad f_\beta(z) = -\frac{1-z^N}{\zeta_\beta(z)} \qquad \text{in the first case,}$$

$$(ii) \qquad f_\beta(z) = -\frac{1}{\zeta_\beta(z)} \qquad \text{in the second case,}$$

and, denoting by $t_1, t_2, \ldots \in \{0,1\}$ the coefficients in

$$-1 + t_1 z + t_2 z^2 + t_3 z^3 + \ldots = f_\beta(z) = -(1-\beta z)\Big(\sum_{n=0}^{\infty} T_\beta^n(1)\, z^n\Big),$$

$f_\beta(z)$ is such that $0.t_1 t_2 t_3 \ldots$ is the **Rényi β-expansion of unity $d_\beta(1)$**.

The Parry Upper function $f_\beta(z)$ has no zero in $\{z \in \mathbb{C} : |z| \leq 1/\beta\}$ except $z = 1/\beta$ which is a simple zero.

The total ordering $<$ on $(1, +\infty)$ is uniquely in correspondence with the lexicographical ordering $<_{lex}$ on the vector coefficients, i.e. on the Rényi expansions of 1 by (Parry, '60) :

In Rényi β-expansions, **the correspondence** $\beta \leftrightarrow (t_i = t_i(\beta))$ **is a bijection.**

Proposition[Parry] : Let $\alpha > 1$ and $\beta > 1$. If the Rényi α-expansion of 1 is

$$d_\alpha(1) = 0.t'_1 t'_2 t'_3 \ldots, \qquad i.e. \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \ldots$$

and the Rényi β-expansion of 1 is

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots, \qquad i.e. \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \ldots,$$

then $\alpha < \beta$ if and only if $(t'_1, t'_2, t'_3, \ldots) <_{lex} (t_1, t_2, t_3, \ldots)$.

$$\{\theta_n^{-1}; n \geq 3\}, \qquad \text{(seq. of Perron numbers} > 1 \text{ tending to 1)}$$

where
θ_n is the unique root of $-1+x+x^n$ in $(0,1)$.

Theorem

Let $n \geq 2$. A real number $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ belongs to $[\theta_{n+1}^{-1}, \theta_n^{-1})$ if and only if the Rényi β-expansion of unity $d_\beta(1)$ is of the form

$$d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3}\ldots, \tag{5}$$

with $n_k \geq n-1$ for all $k \geq 1$.

Pf. : Since $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$ and $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$, the condition $\beta \in [\theta_{n+1}^{-1}, \theta_n^{-1})$ implies that the condition is sufficient.
It is also necessary : $d_\beta(1)$ begins as $0.10^{n-1}1$ for all β such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such βs we write $d_\beta(1) = 0.10^{n-1}1u$ with digits in the alphabet $\mathscr{A}_\beta = \{0, 1\}$ common to all βs, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2}\ldots$$

and $h_0, n_1, h_1, n_2, h_2, \ldots$ integers ≥ 0. The Conditions of Parry (Parry '60, ref : Frougny, Lothaire) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_3}, \ldots)$, which characterizes uniquely the base of numeration β, readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n-1$ for all $k \geq 1$.

Def. : The polynomials of the class $\mathscr{B}$

are all the polynomial sections of the power series $f_\beta(z)$ for β in the interval $(1,(1+\sqrt{5})/2)$. Indeed, from the above, for

$$\beta \in [\theta_{n+1}^{-1}, \theta_n^{-1}),$$

the power series $f_\beta(x)$ takes the form :

$$-1+x+x^n+x^{m_1}+x^{m_2}+\ldots+x^{m_s}+\ldots$$

with the **distanciation conditions :**

$$m_1-n \geq n-1, m_{q+1}-m_q \geq n-1 \quad \textbf{for } 1 \leq q$$

let θ_n be the unique root of the trinomial $G_n(z) := -1 + z + z^n$ in $(0, 1)$.

α	$f_\alpha(z)$
θ_n^{-1}	$-1 + z + z^n$
$\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$	$(-1 + x + x^n) + (x^{m_1} + x^{m_2} + \ldots + x^{m_s} + \ldots)$
	where $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$
θ_{n-1}^{-1}	$-1 + z + z^{n-1}$

Def. : n is called the **dynamical degree of β**, denoted by $\mathrm{dyg}(\beta)$.

The **Parry Upper function** at β

$\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ $\qquad -1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s} + \ldots$

satisfies (Conditions of admissibility of Parry) :
$m_1 - n \geq n-1$, $m_{q+1} - m_q \geq n-1$ for $1 \leq q$
[lexicographical ordering implies moderate gappiness]

its zeroes in $D(0,1)$= eigenvalues^{-1} of the transfer operators (generalized Fredholm determinants - BaladiKeller)
= poles of $\zeta_\beta(z)$.

$$\beta \to 1^+ \qquad \Longleftrightarrow \qquad n = \mathrm{dyg}(\beta) \to \infty.$$

Poles of $\zeta_\beta(z)$ in $D(0,1)$: are limits of zeroes of polynomials of the class $\mathscr{B}$ (Hurwitz). In $D(0,1)$, the poles of $\zeta_\beta(z)$ are separated into two subcollections :

the **lenticular poles**,

the non-lenticular poles.

Description of the lenticular zeroes of $f \in \mathscr{B}$:

- leave the comfort of Taylor series (including the formulation as hypergeometric functions as functions of the coefficients, as in Mellin '15),
- enter "Poincaré asymptotic expansions as a function of the dynamical degree $n = \mathrm{dyg}(\beta)$, i.e. the integer controling the lacunarity a minima.
- replace the usual degree deg(β) by the dynamical degree $\mathrm{dyg}(\beta)$.

Classification not canonical : due to the method of Rouché.

From the structure of the asymptotic expansions of the roots of G_n it is natural to restrict the angular sector to

$$-\pi/18 < \arg \omega < +\pi/18.$$

More precisely,

Theorem (VG, '17)

Let $n \geq 260$. There exists two positive constants c_n and $c_{A,n}$, $c_{A,n} < c_n$, such that the roots of $f \in \mathscr{B}_n$,

$$f(x) - 1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, lying in $-\pi/18 < \arg z < +\pi/18$ either belong to

$$\{z \mid ||z| - 1| < \frac{c_{A,n}}{n}\}, \quad \textit{or to} \quad \{z \mid ||z| - 1| \geq \frac{c_n}{n}\}.$$

true even though n is smaller than 260 on many examples.

Definition

Let $n \geq 260$. Let $\beta > 1$ be a real number such that $\mathrm{dyg}(\beta) = n$. The poles of $\zeta_\beta(z)$ which belong to the angular sector

$$\left\{ z \in \mathbb{C} : |z| < 1 - \frac{c_{lent}}{n}, |\arg z| \leq +\frac{\pi}{18} \right\} \tag{6}$$

are called the *lenticular poles* of $\zeta_\beta(z)$.

The lenticuli of lenticular poles, relative to θ_n^{-1} and β respectively, are :

$$\mathscr{L}_{\theta_n^{-1}} := \{\overline{z_{\lfloor n/6 \rfloor,n}}, \ldots, \overline{z_{2,n}}, \overline{z_{1,n}}, \theta_n, z_{1,n}, z_{2,n}, \ldots, z_{\lfloor n/6 \rfloor,n}\},$$

$$\mathscr{L}_\beta = \{\overline{\omega_{J_n,n}}, \ldots, \overline{\omega_{1,n}}, \beta^{-1}, \omega_{1,n}, \ldots, \omega_{J_n,n}\}.$$

Solomyaks's fractal

The lenticular roots lie on (universal) continuous curves stemming from $z = 1$, including the boundary of Solomyak's fractal.

Let

$$\mathscr{W} := \{h(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid a_j \in [0,1]\}$$

be the class of power series defined on $|z| < 1$ equipped with the topology of uniform convergence on compacts sets of $|z| < 1$. Let $\mathscr{W}_{0,1} \subset \mathscr{W}$ denote functions whose coefficients are in $\{0,1\}$. The space $\mathscr{W}$ is compact and convex. Let

$$\mathscr{G} := \{\lambda \mid |\lambda| < 1, \exists\, h(z) \in \mathscr{W} \text{ such that } h(\lambda) = 0\} \quad \subset \ \{z \mid |z| < 1\}$$

The domain $D(0,1) \setminus \mathscr{G}$ is star-convex due to the fact that :

$$h(z) \in \mathscr{W} \Longrightarrow h(z/r) \in \mathscr{W}, \quad \text{for any } r > 1.$$

The zeroes of any $f \in \mathscr{B}$ lie in $\mathscr{G}$ since :
$t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor = \beta T_\beta^{i-1}(1) - T_\beta^i(1)$, for $i \geq 1$, implies the factorization

$$-1 + t_1 x + t_2 x^2 + t_3 x^3 + \ldots = -(1 - \beta x)(1 + \sum_{j \geq 1} T_\beta^i(1) x^i)$$

– –

For every $\phi \in (0, 2\pi)$, there exists $\lambda = re^{i\phi} \in \mathscr{G}$; the point of minimal modulus with argument ϕ is denoted $\lambda_\phi = \rho_\phi e^{i\phi} \in \mathscr{G}$, $\rho_\phi < 1$.

A function $h \in \mathscr{W}$ is called ϕ-optimal if $h(\lambda_\phi) = 0$.

Denote by $\mathscr{K}$ the subset of $(0, \pi)$ for which there exists a ϕ-optimal function belonging to $\mathscr{W}_{0,1}$.

Denote by $\partial \mathscr{G}_S$ the “spike” : $[-1, \frac{1}{2}(1 - \sqrt{5})]$ on the negative real axis.

Theorem (Solomyak, '94)

(i) The union $\mathscr{G} \cup \mathbb{T} \cup \partial \mathscr{G}_S$ *is closed, symmetrical with respect to the real axis, has a cusp at* $z = 1$ *with logarithmic tangency.*

(ii) the boundary $\partial \mathscr{G}$ *is a* **continuous curve**, *given by* $\phi \to |\lambda_\phi|$ *on* $[0, \pi)$, *taking its values in* $[\frac{\sqrt{5}-1}{2}, 1)$, *with* $|\lambda_\phi| = 1$ *if and only if* $\phi = 0$. *It admits a left-limit at* π^-, $1 > \lim_{\phi \to \pi^-} |\lambda_\phi| > |\lambda_\pi| = \frac{1}{2}(-1+\sqrt{5})$, *the left-discontinuity at* π *corresponding to the extremity of* $\partial \mathscr{G}_S$.

(iii) at all points $\rho_\phi e^{i\phi} \in \mathscr{G}$ *such that* ϕ / π *is rational in an* **open dense subset** *of* $(0, 2)$, $\partial \mathscr{G}$ *is non-smooth,*

(iv) there exists a nonempty subset of transcendental numbers L_{tr}, *of Hausdorff dimension zero, such that* $\phi \in (0, \pi)$ *and* $\phi \notin \mathscr{K} \cup \pi\mathbb{Q} \cup \pi L_{tr}$ *implies that the boundary curve* $\partial \mathscr{G}$ *has a tangent at* $\rho_\phi e^{i\phi}$ *(smooth point).*

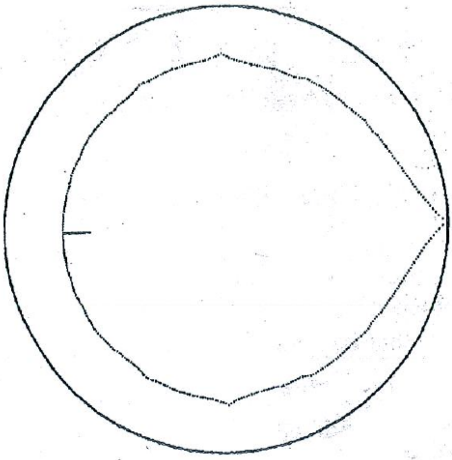


FIGURE: Solomyak's fractal.

Minoration of Mahler measure : Importance of the angular sector $|\arg(z)| < \pi/18$ containing the point 1 was already guessed by :

M. LANGEVIN, *Méthode de Fekete-Szegő et Problème de Lehmer*, C.R. Acad. Sci. Paris Série I Math. **301** (1) (1985), 463–466.
M. LANGEVIN, *Minorations de la Maison et de la Mesure de Mahler de Certains Entiers Algébriques*, C.R. Acad. Sci. Paris Série I Math. **303** (12) (1986), 523–526.
M. LANGEVIN, *Calculs Explicites de Constantes de Lehmer*, in *Groupe de travail en Théorie Analytique et Elémentaire des nombres*, 1986–1987, Publ. Math. Orsay, Univ. Paris XI, Orsay **88** (1988), 52–68.

A. DUBICKAS and C. SMYTH, *The Lehmer Constant of an Annulus*, J. Théorie Nombres Bordeaux **13** (2001), 413–420.

G. RHIN and C.J. SMYTH, *On the Absolute Mahler Measure of Polynomials Having all Zeros in a Sector*, Math. Comp. **64** (1995), 295–304.
G. RHIN and Q. WU, *On the Absolute Mahler Measure of Polynomials Having all Zeros in a Sector II*, Math. Comp. **74** (2005), 383–388.

lenticulus $\mathscr{L}_{\theta_n^{-1}}$ of simple zeroes in $\arg(z) \in (-\pi/3, +\pi/3), n = 71$ and $= 12$.

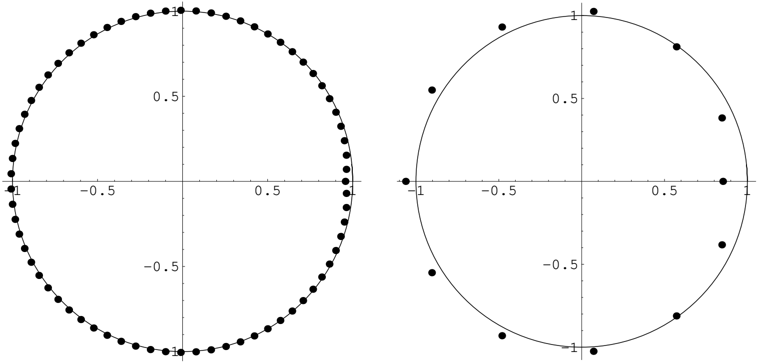


FIGURE: Roots of $G_{71}(z)$, $G_{12}(z)$.

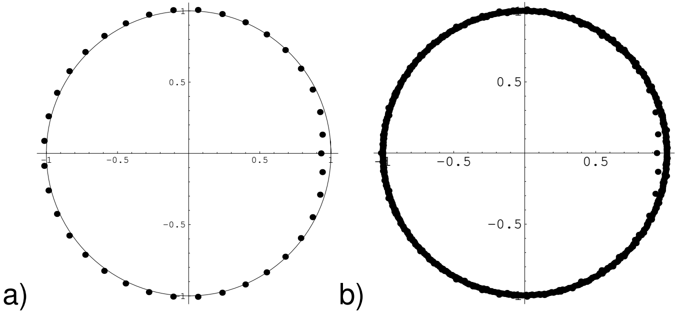


FIGURE: a) The 37 zeroes of $G_{37}(x) = -1 + x + x^{37}$, b) The 649 zeroes of $f(x) = G_{37}(x) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649} = G_{37}(x) + x^{81} + \ldots + x^{649}$. The lenticulus of roots of f (having 3 simple zeroes) is obtained by a very slight deformation of the restriction of the lenticulus of roots of G_{37} to the angular sector $|\arg z| < \pi/18$, off the unit circle. The other roots (nonlenticular) of f can be found in a narrow annular neighbourhood of $|z| = 1$.

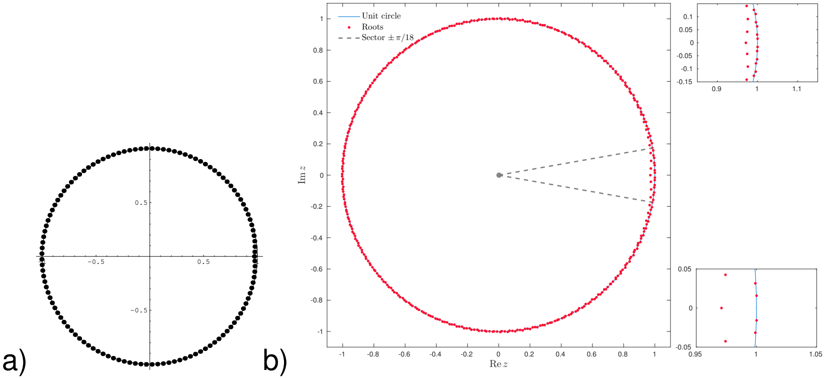


FIGURE: a) Zeroes of G_{121}, b) Zeroes of $f(x) = -1 + x + x^{121} + x^{250} + x^{385}$. On the right the distribution of the roots of f is zoomed twice in the angular sector $-\pi/18 < \arg(z) < \pi/18$. The lenticulus of roots of f has 7 zeroes.

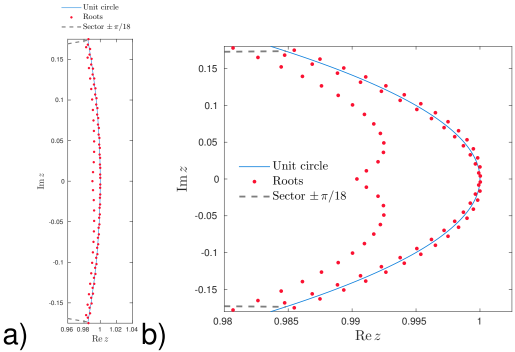


FIGURE: The representation of the 27 zeroes of the lenticulus of $f(x) = -1 + x + x^{481} + x^{985} + x^{1502}$ in the angular sector $-\pi/18 < \arg z < \pi/18$ in two different scalings in x and y (in a) and b)). In this angular sector the other zeroes of f can be found in a thin annular neighbourhood of the unit circle. The real root $1/\beta > 0$ of f is such that β satisfies : $1.00970357\ldots = \theta_{481}^{-1} < \beta = 1.0097168\ldots < \theta_{480}^{-1} = 1.0097202\ldots$.

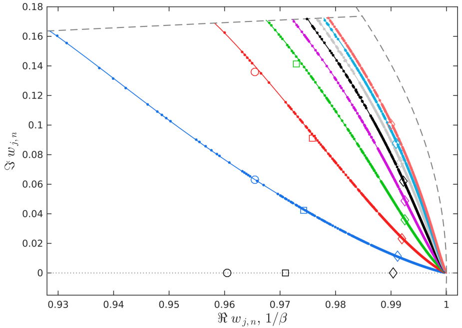


FIGURE: Universal curves stemming from 1 which constitute the lenticular zero locus of all the polynomials of the class $\mathscr{B}$. These curves are continuous, semi-fractal. The first one above the real axis, corresponding to the zero locus of the first lenticular roots, lies in the boundary of Solomyak's fractal [**?**]. The lenticular roots of the previous polynomials *f* are represented by the respective symbols o, $\Box$, $\diamond$. The dashed lines represent the unit circle and the top boundary of the angular sector $|\arg z| < \pi/18$.

A Dobrowolski-type inequality for the lenticular poles

Obj. : asympotic expansions of the lenticular poles as functions of $\mathrm{dyg}(\beta)$ (Mimicked from Poincaré Traité de Mécanique Céleste, vol. 2).

Def. : If $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ for some n large enough, and

$$\mathscr{L}_\beta = \{\overline{\omega_{J_n,n}}, \ldots, \overline{\omega_{1,n}}, \beta^{-1}, \omega_{1,n}, \ldots, \omega_{J_n,n}\}$$

denotes the set of the lenticular poles of $\zeta_\beta(z)$, all depending upon β, then we attribute to $\mathscr{L}_\beta$ a measure, that we call *lenticular measure* of β, by the expression

$$\mathfrak{S}_{lent}(\beta) := \prod_{\omega \in \mathscr{L}_\beta} |\omega|^{-1} = \beta \prod_{j=1}^{J_n} |\omega_{j,n}|^{-2}. \tag{7}$$

By construction, $\mathfrak{S}_{lent}(\beta) \geq 1$.
If $\beta = \theta_n^{-1}$, then the identification with the Mahler measure of θ_n^{-1} holds :

$$\mathfrak{S}_{lent}(\theta_n^{-1}) = \mathrm{M}(\theta_n^{-1}).$$

Denote by $a_{\max} = 5.87433\ldots$ the abscissa of the maximum of the function

$$a \mapsto (1 - \exp(\frac{-\pi}{a}))/(2\exp(\frac{\pi}{a}) - 1)$$

on $(0, \infty)$. Let $\kappa := 0.171573\ldots$ be the value of its maximum, at $a = a_{\max}$. From a numerical viewpoint we have : $2\arcsin(\kappa/2) = 0.171784\ldots$.

Denote

$$C := \exp\Big(\frac{-1}{\pi}\int_0^{2\arcsin(\frac{\kappa}{2})} \mathrm{Log}\Big[\frac{1 + 2\sin(\frac{x}{2}) - \sqrt{1 - 12\sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4}\Big]dx\Big)$$

$$= \mathbf{1.15411}\ldots.$$

Theorem (VG '17)

There exists an integer $\eta \geq 260$ such that the following inequality holds :

$$\mathfrak{S}_{lent}(\beta) \geq C - C\frac{\arcsin(\kappa/2)}{\pi}\frac{1}{\mathrm{Log}\,(n)}, \qquad \text{for all } n \geq \eta$$

and any $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$.

$$n = \mathrm{dyg}(\beta)$$

$$C = 1.15411\ldots$$

This theorem extends (VG'16) :

$$\mathrm{M}(-1+x+x^n) = \mathrm{M}(\theta_n^{-1}) \; > \; \Lambda - \frac{\Lambda}{6}\left(\frac{1}{\mathrm{Log}\, n}\right), \qquad n \geq 2,$$

where Λ is the following constant

$$\Lambda := \exp\left(\frac{3\sqrt{3}}{4\pi}\mathrm{L}(2,\chi_3)\right) \; = \; \exp\left(\frac{-1}{\pi}\int_0^{\pi/3} \mathrm{Log}\left(2\sin\left(\frac{x}{2}\right)\right) dx\right)$$

$$= \; \mathbf{1.38135}\ldots,$$

higher than $C = 1.1541\ldots$, and $L(s,\chi_3) := \sum_{m\geq 1}\frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3, with χ_3 the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0$, 1 or 2 $(\mathrm{mod}\ 3)$, equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol).

Contents

Hyp. : existence of a reciprocal algebraic integer β in the interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$ for some integer n, large enough, such that

$$\mathrm{M}(\beta) < 1.176280\ldots,$$

<u>**Proposition**</u> : for any lenticular pole $\omega_{j,n}$ of $\zeta_\beta(z)$,

$$f_\beta(\omega_{j,n}) = 0 \qquad \Longrightarrow \qquad P_\beta(\omega_{j,n}) = 0.$$

This identification gives the following minoration to the Mahler measure of β :

$$\mathrm{M}(\beta) = \mathrm{M}(\beta^{-1}) = \prod_{\omega \notin \mathscr{L}_\beta, |\omega|<1} |\omega|^{-1} \times \prod_{\omega \in \mathscr{L}_\beta} |\omega|^{-1} \geq \prod_{\omega \in \mathscr{L}_\beta} |\omega|^{-1} = \mathfrak{S}_{lent}(\beta).$$

Pf. : rewriting trails.

Two Theorems : Factorization of the polynomials of the class $\mathscr{B}$ and Kala-Vávra's Theorem

Theorem (Dutykh - VG '18)

For any $f \in \mathscr{B}_n$, $n \geq 3$, denote by

$$f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where A is the cyclotomic part, B the reciprocal noncyclotomic part, C the nonreciprocal part. Then

(i) the nonreciprocal part C is nontrivial, irreducible and **never vanishes on the unit circle***,*

(ii) if $\gamma > 1$ denotes the real algebraic number uniquely determined by the sequence $(n, m_1, m_2, \ldots, m_s)$ such that $1/\gamma$ is the unique real root of f in (θ_{n-1}, θ_n), $-C^*(X)$ is the minimal polynomial $P_\gamma(X)$ of γ, and γ is a nonreciprocal algebraic integer.

Pf. : Ljunggren's tricks

For a general complex number $\beta \in \mathbb{C}$, $|\beta| > 1$, and a finite alphabet $\mathscr{A} \subset \mathbb{C}$, we define the $(\beta, \mathscr{A})$-representations as expressions of the form

$$\sum_{k \geq -L} a_k \beta^{-k}, \qquad a_k \in \mathscr{A},$$

for some integer $L \in \mathbb{Z}$. They are Laurent series of $1/\beta$. We define

$$\mathrm{Per}_{\mathscr{A}}(\beta) := \{x \in \mathbb{C} \mid x \text{ has an eventually periodic } (\beta, \mathscr{A})\text{−representation}\}.$$

Those βs which are the real roots > 1 of the polynomials of the class $\mathscr{B}$ will be of special interest in the next section.

Theorem (Kala -Vávra '19)

Let $\beta \in \mathbb{C}$ be an algebraic number of degree d, $|\beta| > 1$, and let $a_d x^d - a_{d-1} x^{d-1} - \ldots - a_1 x - a_0 \in \mathbb{Z}[x]$ be its minimal polynomial. **Suppose that $|\beta'| \neq 1$ for any conjugate β' of β.** *Then there exists a finite alphabet $\mathscr{A} \subset \mathbb{Z}$ such that*

$$\mathbb{Q}(\beta) = \mathrm{Per}_{\mathscr{A}}(\beta). \tag{8}$$

Rewriting polynomials - identification of lenticular roots as conjugates [Res. Number Theory 7 :64 (2021)]

Take $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$ a **reciprocal** algebraic integer for some integer n large enough. Then

$$f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + \ldots + z^{m_j} + z^{m_{j+1}} + \ldots,$$

where $m_1 - n \geq n-1$, $m_{j+1} - m_j \geq n-1$ for $j \geq 1$, written $= -1 + \sum_{i\geq 1} t_i z^i$, is a power series which is never a polynomial (Descartes's rule).

For every $s \geq 1$, let $S_{\gamma_s}(X) := X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$ such that

$$S^*_{\gamma_s}(X) = X^s S_{\gamma_s}(1/X) = 1 - t_1 X - t_2 X^2 - \ldots - t_{s-1} X^{s-1} - t_s X^s$$

and $-S^*_{\gamma_s}(z)$ is the sth polynomial section of $f_\beta(z)$.

Def. : $\gamma_s > 1$ unique zero of $S_{\gamma_s}(z)$. : $S^*_{\gamma_s}(\gamma_s^{-1}) = 0$, $\lim_{s\to+\infty} \gamma_s^{-1} = \beta^{-1}$.

The minimal polynomial P_β of β is monic and reciprocal. Denote it

$$P_\beta(x) = 1 + a_1 x + a_2 x^2 + \ldots + a_{d-1}x^{d-1} + a_d x^d \qquad (a_{d-i} = a_i).$$

Let $H := \max\{|a_i| : 1 \leq i \leq d-1\} \geq 1$ be the naïve height of P_β.

We have 2 “characterizers” of the same object β :

P_β **and** f_β**.**

How to pass from one to the other ? rewriting trail and control of the alphabet in Kala-Vávra’s Theorem.

We proceed in two steps :

(i) first we express $P_\beta(\gamma_s)$ as a $(\gamma_s, \mathscr{A})$- eventually periodic representation with the symmetrical alphabet $\mathscr{A} = \{-m, \ldots, 0, \ldots, +m\} \subset \mathbb{Z}$ which does not depend upon s, with $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$. This expression of $P_\beta(\gamma_s)$, as a Laurent series of $1/\gamma_s$, is obtained using the two above Theorems ,

(ii) we allow s to tend to infinity to obtain the convergence of $P_\beta(\sigma(\gamma_s))$ to 0 which will imply $P_\beta(\sigma(\beta)) = 0$.

Starting point : $1 = 1$,
to which we add $0 = -S_{\gamma}^{*}(\gamma^{-1})$ in the right hand side. Then we define a rewriting trail from

$$1 = 1 - S_{\gamma}^{*}(\gamma^{-1}) = t_1\gamma^{-1} + t_2\gamma^{-2} + \ldots + t_{s-1}\gamma^{-(s-1)} + t_s\gamma^{-s}$$

to

$$-a_1\gamma^{-1} - a_2\gamma^{-2} + \ldots - a_{d-1}\gamma^{-(d-1)} - a_d\gamma^{-d} = 1 - P(\gamma^{-1}).$$

A rewriting trail will be a sequence of integer polynomials, whose role will consist **in "restoring" the coefficients** of $1 - P(\gamma^{-1})$ one after the other, from the left, by adding "0" conveniently at each step to both sides.

At the first step we add $0 = (-a_1 - t_1)\gamma^{-1} S_{\gamma}^{*}(\gamma^{-1})$; and we obtain

$$1 = -\mathbf{a_1}\gamma^{-1}$$

$$+(-(-a_1 - t_1)t_1 + t_2)\gamma^{-2} + (-(-a_1 - t_1)t_2 + t_3)\gamma^{-3} + \ldots$$

so that the height of the polynomial

$$(-(-a_1 - t_1)t_1 + t_2)X^2 + (-(-a_1 - t_1)t_2 + t_3)X^3 + \ldots$$

is $\leq H + 2$.

At the second step we add $0 = (-a_2 - (-(-a_1 - t_1)t_1 + t_2))\gamma^{-2} S^*_\gamma(\gamma^{-1})$. Then we obtain

$$1 = -\mathbf{a_1}\gamma^{-1} - \mathbf{a_2}\gamma^{-2}$$
$$+[(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]\gamma^{-3} + \ldots$$

where the height of the polynomial

$$[(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]X^3 + \ldots$$

is $\leq H + (H+2) + (H+2) = 3H + 4$. Iterating this process d times we obtain

$$1 = -\mathbf{a_1}\gamma^{-1} - \mathbf{a_2}\gamma^{-2} - \ldots - \mathbf{a_d}\gamma^{-d}$$
$$+ \ \textit{polynomial remainder in } \gamma^{-1}.$$

To summarize, we obtain a sequence $(A'_q(X))_{q\geq 1}$ of rewriting polynomials involved in this rewriting trail ; for $q \geq 1$, $A'_q \in \mathbb{Z}[X]$, $\deg(A'_q) \leq q$ and $A'_q(0) = -1$. The first polynomial $A'_1(X)$ is $-1 + (-a_1 - t_1)X$. The second polynomial $A'_2(X)$ is $-1 + (-a_1 - t_1)X + (-a_2 - (-(-a_1 - t_1)t_1 + t_2))X^2$, etc.

Contents

Theorem

There exists an integer $\eta \geq 260$ such that, for any nonzero real reciprocal algebraic integer β, which is $\neq \pm 1$,

$$\mathrm{M}(\beta) \geq \theta_\eta^{-1}.$$

$\theta_{260}^{-1} \approx 1.01\ldots$

Pf. : Let $\beta \neq 0$ be a reciprocal algebraic integer which is not a root of unity, such that $\mathrm{dyg}(\beta) \geq \eta$ with $\eta \geq 260$. Since $\mathrm{M}(\beta) = \mathrm{M}(\beta^{-1})$ there are three cases to be considered :

(i) the house of β satisfies $\overline{|\beta|} \geq \theta_5^{-1}$,

(ii) the dynamical degree of β satisfies : $6 \leq \mathrm{dyg}(\beta) < \eta$, with $\mathrm{M}(\beta) < 1.176280\ldots$,

(iii) the dynamical degree of β satisfies : $\mathrm{dyg}(\beta) \geq \eta$, with $\mathrm{M}(\beta) < 1.176280\ldots$.

In the first case, $\mathrm{M}(\beta) \geq \theta_5^{-1} > \theta_{260}^{-1} \geq \theta_\eta^{-1} > 1$. In the second case, $\mathrm{M}(\beta) \geq \theta_\eta^{-1}$. In the third case, the **Dobrowolski-type inequality** gives the following lower bound of the Mahler measure

$$\mathrm{M}(\beta) \geq C - C\frac{\arcsin(\kappa/2)}{\pi \,\mathrm{Log}\,(\mathrm{dyg}(\beta))}$$

$$\geq C - C\frac{\arcsin(\kappa/2)}{\pi \,\mathrm{Log}\,(\eta)} \geq C - C\frac{\arcsin(\kappa/2)}{\pi \,\mathrm{Log}\,(260)}, \approx 1.14\ldots$$

This lower bound is numerically greater than $\theta_{260}^{-1} = 1.01\ldots$, itself greater than θ_η^{-1}. In any case, the universal lower bound θ_η^{-1} of $\mathrm{M}(\beta)$ holds true.